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LETTER TO THE EDITOR

Non-linear electron plasma waves

Dipankar Ray
Physics Department, New York University, New York, NY 10003, USA

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Abstract. An approximate solution for equations for non-linear electron plasma waves in an unmagnetised plasma has been given recently by Shukla and Tagare. It is shown that their approximation is not quite valid and a more rigorous study is made.

1. Introduction

In a recent paper, Shukla and Tagare (1977) have obtained the following equation for non-linear electron plasma waves in an unmagnetised plasma:

$$\left(\frac{U^2}{(1+N)^3}-\frac{1}{1+N}\right)N_{\xi\xi}+\left(\frac{1}{(1+N)^2}-\frac{3U^2}{(1+N)^4}\right)(N_\xi)^2+N=0 \tag{1.1}$$

where

$$U=V/v_e, \qquad n=1+N, \qquad \xi=x-Ut \quad (U \text{ is a constant}), \tag{1.2}$$

and $N\equiv N(\xi)$, $N_\xi=\mathrm{d}N/\mathrm{d}\xi$, $N_{\xi\xi}\equiv\mathrm{d}^2N/\mathrm{d}\xi^2$ and so on. n is the electron density, normalised by average particle number density, V is the velocity of the non-linear wave in the moving frame and v_e, the electron thermal velocity.

According to Shukla and Tagare (1977), (1.1) reduces to

$$(U^2-1)N_{\xi\xi}-(3U^2-1)(N_\xi)^2+N=0 \tag{1.3}$$

$$\text{for } |N|\ll 1. \tag{I}\dagger$$

Equation (1.3) can be rewritten as

$$2H_{\zeta\zeta}-(H_\zeta)^2+H=0 \tag{1.4}$$

provided $U>1$ and

$$H=\frac{2(3U^2-1)}{U^2-1}N, \qquad \zeta=\left(\frac{2}{(U^2-1)}\right)^{1/2}\xi, \qquad H_\zeta\equiv\frac{\mathrm{d}H}{\mathrm{d}\zeta}, \qquad H_{\zeta\zeta}\equiv\frac{\mathrm{d}^2H}{\mathrm{d}\rho^2}.$$

Equation (1.4) has known periodic solutions (Alterkop and Rukhadze 1972) with known oscillation period and other characteristics. Shukla and Tagare have argued that from these known solutions of (1.4), an approximate idea of the solutions of (1.1) can be obtained.

† Throughout this letter, equations are denoted by arabic numerals and inequalities by roman numerals.

However, the fact that the approximated equation, i.e. equation (1.3), admits periodic solutions for N as a function of ξ, does not necessarily mean that the orginal equation, i.e. equation (1.1), will also have periodic solutions. Moreover, since the validity of approximation depends on the value of N, which can be determined only by solving (1.1) itself, the above mentioned work does not provide enough clues to certain knowledge as to when the approximation is valid†. Thus it appears to me that it would be worthwhile to attempt a more rigorous study.

2. Solution of the differential equation

Although Shukla and Tagare have considered the case of $U > 1$, only, we do not intend to impose such a restriction at the outset. However, it is easy to see that without loss of generality one can set

$$U > 0. \tag{II}$$

Setting

$$W \equiv U^2/(1+N)^3 - 1/(1+N) \tag{2.1}$$

(1.1) can be rewritten as

$$WN_{\xi\xi} + (\mathrm{d}W/\mathrm{d}N)(N_\xi)^2 + N = 0$$

or

$$(W/2)(\mathrm{d}/\mathrm{d}N)(N_\xi^2) + (\mathrm{d}W/\mathrm{d}N)(N_\xi^2) + N = 0. \tag{2.2}$$

Using $2W$ as integrating factor (2.2) can be integrated to give

$$W^2 N_\xi^2 + V = A \tag{2.3}$$

where A is a constant, W is given by (2.1) and

$$V \equiv 2\{U^2[1/2(1+N)^2 - 1/(1+N)] - N + \ln(1+N)\}. \tag{2.4}$$

It should be noted that in view of (1.2) one must have $1 + N \geqslant 0$. However, from (2.1) and (2.4) we note that as $N + 1 \to 0$, $W \to \infty$ and $V \to \infty$. Thus (2.3) cannot hold for $1 + N = 0$ and we get

$$1 + N > 0. \tag{III}$$

From (2.3) one can easily write

$$\int \frac{W\,\mathrm{d}N}{(A-V)^{1/2}} = \pm\xi \tag{2.5}$$

where, as before, W and V are given by (2.1) and (2.4) and A is a constant.

† It should also be noted that the validity of the approximation does not depend on the value of N only, as might appear from (I), which is given by Shukla and Tagare (1977), as a condition for validity of the approximation. The validity of the approximation depends on the values of both U and N; e.g. no matter how small N is, if $U^2/(1+N)^3 - 1/(1+N)$ and $U^2 - 1$ have different signs, then it is obvious from comparison of (1.1) and (1.3) that the two equations give entirely different types of curves for those values of N and the approximation is not valid.

3. Physical constraints

Equation (2.5) gives all the solutions of (1.1). However, not all of them are physically meaningful. A physically meaningful solution must give N as a single-valued function of ξ and N and N_ξ must be finite everywhere. Which members of the family of curves (2.5) satisfy these conditions is seen as follows.

Since N is finite everywhere, let N_1 and N_2 respectively be the greatest lower bound and least upper bound of N‡.

Then in view of (III)

$$-1 < N_1 < N_2 \tag{IV}$$

and $N_\xi \to 0$ as $N \to N_1, N_2$ and since in view of (IV), W is finite at $N = N_1, N_2$, we get from (2.3)

$$V = A \qquad \text{at } N = N_1, N_2. \tag{3.1}$$

From (3.1), using Roll's theorem, we get that there exists at least one N_0, such that

$$N_1 < N_0 < N_2 \tag{V}$$

and

$$\mathrm{d}V/\mathrm{d}N = 0 \qquad \text{for } N = N_0. \tag{3.2}$$

We shall show that N_0 is unique and

$$N_0 = 0. \tag{3.3}$$

From (2.4), we note that there are only two values of N consistent with (III) for which $\mathrm{d}V/\mathrm{d}N = 0$. They are $N = 0$ and $N = U - 1$. However, $U - 1$ must be outside the interval $[N_1, N_2]$, as can be seen as follows.

If possible, let $N_1 < U - 1 < N_2$. From (2.1), $W = 0$ at $N = U - 1$. Therefore from (2.3), $V = A$ at $N = U - 1$. Then from (3.1) and Roll's theorem, there exists N', N'' such that

$$\mathrm{d}V/\mathrm{d}N = 0 \qquad \text{for } N = N', N''$$

and

$$N_1 < N' < U - 1 < N'' < N_2,$$

i.e. there are three values of N, namely $N', U - 1, N''$, satisfying (III) for which $\mathrm{d}V/\mathrm{d}N = 0$; this is impossible since we have already seen that there are only two such values of N, namely 0 and $U - 1$. Thus $N = 0$ is the only value of N for which $\mathrm{d}V/\mathrm{d}N = 0$ and which can belong to the interval $[N_1, N_2]$. Thus (3.3) follows.

From (2.3) it is easy to note that, in the interval $[N_1, N_2]$, $V \leqslant A$. Since in $[N_1, N_2]$, we get $\mathrm{d}V/\mathrm{d}N = 0$, only at $N = 0$, there must be a minimum for V at $N = 0$, i.e.

$$\mathrm{d}^2V/\mathrm{d}N^2 \geqslant 0 \qquad \text{at } N = 0.$$

‡ An upper bound U of N is a number such that all values taken by N are greater than or equal to U. The least upper bound is the least of such upper bounds. If N has a maximum value, then obviously that maximum value is the least upper bound of N. However, it is possible that N is bounded above, but does not have a maximum value. In that case N must asymptotically approach the least upper bound as $\xi \to \infty$ or as $\xi \to -\infty$. For such cases we say that the least upper bound N_2 is not attained by N. Similar definitions apply to lower bounds and the greatest lower bound.

From (2.4), it is easy to see then that.

$$U > 1. \tag{VI}$$

Combining (IV), (V), (VI), using (3.3) and the fact that $U-1$ lies outside the interval $[N_1, N_2]$, we get

$$-1 < N_1 < 0 < N_2 < U-1. \tag{VII}$$

Thus summing up, we have the following situation:

(i) the system is governed by equation (2.5); W and V are given by (2.1), (2.4).

(ii) it remains bounded in $N_1 \leqslant N \leqslant N_2$, where $A = V(N_1) = V(N_2)$;

(iii) N_1, N_2, U are such that (VII) holds.

However, whether the bounds N_1 and N_2 are attained by N for finite values of ξ will depend on whether or not $\int_{N_1}^{N_2} W\,\mathrm{d}N/(A-V)^{1/2}$ is finite.

We note that the integrand $W/(A-V)^{1/2}$ is finite everywhere except at $N = N_1$ and $N = N_2$. As $N \to N_1$, $W/(A-V)^{1/2} \to \infty$ of the order of $(N-N_1)^{-1/2}$. The same applies to N_2. Noting that, $\int x^{-1/2}\,\mathrm{d}x = 2x^{1/2}$, it can be easily shown that $\int_{N_1}^{N_2} W\,\mathrm{d}N/(A-V)^{1/2}$ is finite. Thus the lower and upper bounds N_1 and N_2 are attained by N for finite values of ξ and N oscillates between these two values, increases from N_1 to N_2, then decreases from N_2 to N_1 and so on. When the value of N increases from N_1 to N_2 with increases of ξ, it satisfies an equation,

$$\int_{N_1}^{N} W\,\mathrm{d}N/(A-V)^{1/2} = \xi + \text{constant}. \tag{3.4}$$

On the other hand when N decreases from N_2 to N_1 with increases of ξ, it satisfies

$$\int_{N_2}^{N} W\,\mathrm{d}N/(A-V)^{1/2} = -\xi + \text{constant} \tag{3.5}$$

and the period of one oscillation is

$$T = 2\int_{N_1}^{N_2} W\,\mathrm{d}N/(A-V)^{1/2}. \tag{3.6}$$

The constants of integration in (3.4) and (3.5) are subject to the restriction that N must be a single-valued function of ξ. Thus if at $\xi = 0$, $N = N_1$, then in general, we can write

$$rT + \int_{N_1}^{N} W\,\mathrm{d}N/(A-V)^{1/2} = \xi \qquad \text{for } rT < \xi < (r+\tfrac{1}{2})T \tag{3.7}$$

$$(r+\tfrac{1}{2})T - \int_{N_1}^{N} W\,\mathrm{d}N/(A-V)^{1/2} = \xi \qquad \text{for } (r+\tfrac{1}{2})T < \xi < (r+1)T \tag{3.8}$$

where r is any integer, T is given by (3.6). N_1, N_2, U satisfy (VII); W, V are given by (2.1) and (2.4); and

$$A = V(N_1) = V(N_2). \tag{3.9}$$

From (VII), (3.9) and the fact that for $U > 1$, V is a monotonic increasing function of N, between 0 and $U-1$, it is easy to see that A must lie between $V_{\text{at}\,N=0}$ and

$V_{\text{at } N=U-1}$, i.e.

$$-U^2 < A < (3 - 4U + 2\ln U) \quad \text{(VIII)}$$

where

$$U > 1. \quad \text{(VI)}$$

4. Conclusion

Thus, equation (2.3) with W and V given by (2.1) and (2.4) gives oscillatory motion for N as a function of ξ if (VI) and (VIII) hold. The oscillatory motion is described by (3.7) and (3.8) where N_1 and N_2 are determined from (3.9) and (VII). T, the period of oscillation, is given by (3.6).

It is to be noted that although we have not been able to solve (3.9) explicitly for N_1, N_2, nor have we been able to explicitly obtain the integrals in (3.6), (3.7) and (3.8), equation (3.9) can be solved numerically for N_1, N_2 with some degree of accuracy and once N_1, N_2 are obtained, integrals in (3.6), (3.7) and (3.8) can be obtained numerically with unlimited accuracy. Thus, although we do not have the exact solution, we have a procedure by which we can be as accurate as we want.

To check the validity of the approximation of (1.1) by (1.3) as done by Shukla and Tagare (1977), we note that the inequality (I), in the present context, means that

$$|N_1| \ll 1, \qquad |N_2| \ll 1. \quad \text{(IX)}$$

However, (IX) is not enough for the validity of the above mentioned approximation. In view of (VII), we also need

$$|N_1| \ll U - 1, \qquad |N_2| \ll U - 1 \quad \text{(X)}$$

e.g. even when (IX) holds, if $N_2 \simeq U - 1$, then near $N = N_2$, the coefficient of $N_{\xi\xi}$ in (1.1), i.e. $U^2/(1+N)^3 - 1/(1+N)$, and the coefficient of $N_{\xi\xi}$ in (1.3), i.e. $U^2 - 1$, can be quantities of different orders of magnitude and the approximation will not be justified.

References

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